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THE C^*-ALGEBRAS OF ALMOST COMMUTING ISOMETRIES ARE GENERALIZED RFD

TAKAHIRO SUDO

Department of Mathematical Sciences
Faculty of Science
University of the Ryukyus
Nishihara
Okinawa 903-0213, Japan
email: sudo@math.u-ryukyu.ac.jp

Abstract

We consider the structure of the C^*-algebras generated by almost commuting isometries and show that they are generalized RFD in a sense.

1. Introduction

The C^*-algebras generated by almost commuting unitaries are introduced by Exel ([4] and [5]) and their structure and K-theory are considered by him. Furthermore, it is shown by Eilers and Exel [3] that the C^*-algebras are RFD, i.e., residually finite dimensional, following a seminal paper of Choi [1], in which it is shown that the full group C^*-algebra of the free group with two generators is RFD. On the other hand, the C^*-algebras generated by almost commuting isometries and their K-theory are studied by the author [7].

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In this paper, following [3] we show that the C^*-algebras of almost commuting isometries are generalized RFD (see the definition below). This suggests that the class of generalized RFD C^*-algebras should be interesting and would be important.

2. The Soft Toeplitz Tensor Product

Definition 1.1. Let $C(\mathbb{T}^2)$ be the universal C^*-algebra generated by two commuting unitaries. It is also the C^*-algebra of all continuous functions on the 2-torus $\mathbb{T}^2$. For $\varepsilon \in [0, 2]$, the soft torus (of Exel) denoted by $C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T})$ is defined to be the universal C^*-algebra generated by two unitaries $u_{\varepsilon,1}, u_{\varepsilon,2}$ such that $\|u_{\varepsilon,2}u_{\varepsilon,1} - u_{\varepsilon,1}\ u_{\varepsilon,2}\| \leq \varepsilon$.

It is shown in [3] that $C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T})$ is RFD, i.e., residually finite dimensional in the sense that for any nonzero element $a \in C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T})$, there exists a finite dimensional representation ρ of $C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T})$ such that $\rho(a) \neq 0$, in other words, there are separating finite dimensional representations of $C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T})$. It is clear that $C(\mathbb{T}^2)$ is RFD.

Definition 1.2. Let $\mathfrak{F}$ be the universal C^*-algebra generated by a proper isometry. It is called Toeplitz algebra. For $\varepsilon \in [0, 2]$, the soft Toeplitz tensor product denoted by $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ is defined to be the universal C^*-algebra generated by two isometries $s_{\varepsilon,1}, s_{\varepsilon,2}$ such that $\|s_{\varepsilon,2}s_{\varepsilon,1} - s_{\varepsilon,1}s_{\varepsilon,2}\| \leq \varepsilon$. There exists the canonical quotient map from $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ to $C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T})$ by sending $s_{\varepsilon,j}$ to $u_{\varepsilon,j}$ $(j = 1, 2)$.

It is known (cf. [6]) that $\mathfrak{F}$ is decomposed into the short exact sequence:

$$0 \to \mathbb{K} \to \mathfrak{F} \to C(\mathbb{T}) \to 0,$$

where $\mathbb{K}$ is the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space. It is easy to see that $\mathfrak{F}$ is not RFD since $\mathbb{K}$ has only one irreducible infinite dimensional representation, i.e., the identity representation.

Definition 1.3. We say that a C^*-algebra is generalized RFD if it has a (finite) composition series of closed ideals such that its subquotients are either RFD or AL, where an AL algebra means an inductive limit of liminary (or CCR) C^*-algebras.

Remark. We may replace being AL with being AI, where an AI algebra means an inductive limit of type I C^*-algebras. In general, a C^*-algebra is of type I if and only if it has a composition series whose subquotients are CCR (see[2]).

By definition, it follows that the Toeplitz algebra $\mathfrak{F}$ is generalized RFD.

Theorem 1.4. *The soft Toeplitz tensor product* $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ *is generalized RFD.*

Proof. Recall from [7] that $\mathfrak{F} \otimes_\varepsilon \mathfrak{F} \cong E_\varepsilon \rtimes_{\alpha\varepsilon} \mathbb{N}$ the semigroup crossed product by $\mathbb{N}$ of natural numbers, where E_ε is the C^*-subalgebra of $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ generated by $t_n = s_{\varepsilon,2}^n \, s_{\varepsilon,1}(s_{\varepsilon,2}^*)^n$ for $n \in \mathbb{N}$ and $n = 0$, and the action α_ε by endomorphisms is defined by $\alpha_\varepsilon(t_n) = s_{\varepsilon,2} t_n s_{\varepsilon,2}^*$ for $n \in \mathbb{N}$ and $n = 0$. We have the following exact sequence:

$$0 \to \varinjlim \mathbb{K} \to E_\varepsilon \to \pi(E_\varepsilon) \to 0,$$

where π is the canonical quotient map from E_ε to $\pi(E_\varepsilon) = \mathfrak{B}'_\varepsilon$ by sending t_n to $v_n = u_{\varepsilon,2}^n \, u_{\varepsilon,1}(u_{\varepsilon,2}^*)^n$ for $n \in \mathbb{N}$ and $n = 0$, where $\mathfrak{B}'_\varepsilon$ is the C^*-algebra generated by unitaries v_n for $n \in \mathbb{N}$ and $n = 0$, and it is shown that the kernel of π is isomorphic to $\varinjlim \mathbb{K}$, an inductive limit of $\mathbb{K}$.

Therefore, it is extended by taking crossed products by $\mathbb{N}$ that

$$0 \to \varinjlim \mathbb{K} \rtimes \mathbb{N} \to E_\varepsilon \rtimes \mathbb{N} \to \pi(E_\varepsilon) \rtimes \mathbb{N} \to 0,$$

and $\varinjlim \mathbb{K} \rtimes \mathbb{N} \cong \varinjlim \mathbb{K} \otimes C^*(\mathbb{N}) = \varinjlim \mathbb{K} \otimes \mathfrak{F}$ since the action of $\mathbb{N}$ on $\varinjlim \mathbb{K}$ is trivial. Furthermore, following the methods of Eilers and Exel [3] we can show that $\pi(E_\varepsilon)$ is RFD, from which for any nonzero element x of $\pi(E_\varepsilon) \rtimes \mathbb{N}$, there exists a quotient map ρ from $\pi(E_\varepsilon) \rtimes \mathbb{N}$ to $M_l(\mathbb{C}) \rtimes \mathbb{N}$ for some l, and the composition of ρ with the canonical expectation from $M_l(\mathbb{C}) \rtimes \mathbb{N}$ to $M_l(\mathbb{C})$ maps x nonzero. It follows that $\rho(x) \neq 0$. Since $M_l(\mathbb{C}) \rtimes \mathbb{N} \cong M_l(\mathbb{C}) \otimes C^*(\mathbb{N})$ with $C^*(\mathbb{N}) \cong \mathfrak{F}$, further taking a quotient map from $M_l(\mathbb{C}) \otimes C^*(\mathbb{N})$ to $M_l(\mathbb{C})$ implies that $\pi(E_\varepsilon) \rtimes \mathbb{N}$ is RFD. Note that

$$0 \to \varinjlim \mathbb{K} \otimes \mathbb{K} \to \varinjlim \mathbb{K} \otimes \mathfrak{F} \to \varinjlim K \otimes C(\mathbb{T}) \to 0,$$

and $\varinjlim \mathbb{K} \otimes \mathbb{K} \cong \varinjlim (\mathbb{K} \otimes \mathbb{K})$ and $\varinjlim \mathbb{K} \otimes C(\mathbb{T}) \cong \varinjlim (\mathbb{K} \otimes C(\mathbb{T}))$ are AL. Hence we obtain the conclusion.

Remark. Note also that $C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T})$ is RFD by [3] and we have the quotient map from $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ to $C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T})$ by sending $s_{\varepsilon, j}$ to $u_{\varepsilon, j}$ for $j = 1, 2$. The conclusion above might follow from considering this map and its kernel part.

3. The Soft Tori and the Soft Pointed Tori

Definition 2.1. For $\varepsilon \in [0, 2]$, the soft n-torus denoted by $\otimes_\varepsilon^n C(\mathbb{T})$ is defined to be the universal C^*-algebra generated by n unitaries $u_{\varepsilon, j}$ $(1 \leq j \leq n)$ such that $\|u_{\varepsilon, k} u_{\varepsilon, j} - u_{\varepsilon, j} u_{\varepsilon, k}\| \leq \varepsilon$ for $1 \leq j, k \leq n$.

Definition 2.2. For $\varepsilon \in [0, 2]$, the soft n-pointed torus denoted by $\otimes_\varepsilon^n \mathfrak{F}$ is defined to be the universal C^*-algebra generated by n isometries $s_{\varepsilon, j}$ $(1 \leq j \leq n)$ such that $\|s_{\varepsilon, k} s_{\varepsilon, j} - s_{\varepsilon, j} s_{\varepsilon, k}\| \leq \varepsilon$ for $1 \leq j, k \leq n$.

Remark. Note that the spectrum of $\mathfrak{F}$ is identified with the union of $\mathbb{T}$ and a point, i.e., a pointed torus, from its exact sequence given in Section 2.

Proposition 2.3. *The soft n-torus is written as the crossed product of a* C^**-algebra by a multi-shift of the group* $\mathbb{Z}$ *of integers.*

The soft n-pointed torus is written as the crossed product of a C^**-algebra by a multi-shift of the semigroup* $\mathbb{N}$ *of natural numbers.*

Proof. First of all, $C(\mathbb{T}) \otimes_\varepsilon C(\mathbb{T}) \cong \mathfrak{B}_\varepsilon \rtimes \mathbb{Z}$ by [4] using universality, where $\mathfrak{B}_\varepsilon$ is the universal C^*-algebra generated by unitaries v_n $(n \in \mathbb{Z})$ such that $\|v_{n+1} - v_n\| \le \varepsilon$, and the action of $\mathbb{Z}$ is the shift on these unitaries. Note that v_n is identified with $u_{\varepsilon,2}^n u_{\varepsilon,1}(u_{\varepsilon,2}^*)^n$ and the action of $\mathbb{Z}$ is in fact the adjoint action by $u_{\varepsilon,2}$.

By the method above, for $1 \le j \le n-1$, the C^*-subalgebra generated by two unitaries $u_{\varepsilon,j}$ and $u_{\varepsilon,n}$ in $\otimes_\varepsilon^n C(\mathbb{T})$ is written as $\mathfrak{B}_{\varepsilon,j} \rtimes \mathbb{Z}$. Extending the action of $\mathbb{Z}$ to (a multi-shift on) the C^*-algebra generated by $\mathfrak{B}_{\varepsilon,j}$ $(1 \le j \le n-1)$, we obtain the first conclusion.

Similarly, the second conclusion follows. Indeed, $\mathfrak{F} \otimes_\varepsilon \mathfrak{F} \cong E_\varepsilon \rtimes \mathbb{N}$, where E_ε is the C^*-algebra generated by the elements $s_{\varepsilon,2}^n s_{\varepsilon,1}(s_{\varepsilon,2}^*)^n$ $(n \in \mathbb{N} \text{ and } n = 0)$ and the action of $\mathbb{N}$ is the adjoint action by $s_{\varepsilon,2}$ on the elements (cf. [7]).

Theorem 2.4. *The soft n-torus is RFD and the soft n-pointed torus is generalized RFD.*

Proof. By the proposition above, $\otimes_\varepsilon^n C(\mathbb{T}) \cong \mathfrak{B}_\varepsilon^{n-1} \rtimes \mathbb{Z}$, where $\mathfrak{B}_\varepsilon^{n-1}$ is the C^*-algebra generated by $\mathfrak{B}_{\varepsilon,j}$ $(1 \le j \le n-1)$. By (extensive) induction, it follows that $\mathfrak{B}_\varepsilon^{n-1}$ is RFD. Indeed, $\mathfrak{B}_\varepsilon^{n-1}$ is generated by C^*- subalgebras $\mathfrak{B}_{\varepsilon,n-1}$ and $\mathfrak{B}_\varepsilon^{n-2}$ that is RFD by induction, which is also RFD. Therefore, it follows that $\mathfrak{B}_{\varepsilon}^{n-1} \rtimes \mathbb{Z}$ is RFD by the same argument as in Section 2.

From the above proposition, $\otimes_{\varepsilon}^{n} \mathfrak{F} \cong E_{\varepsilon}^{n-1} \rtimes \mathbb{N}$, where E_{ε}^{n-1} is defined as $\mathfrak{B}_{\varepsilon}^{n-1}$, so that that the second claim is proved as shown in Theorem 1.4.

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